

ON GENERALIZED (p, q) -ELLIPTIC INTEGRALS

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ABSTRACT. In this paper authors study the generalized (p, q) -elliptic integrals of the first and second kind in the mean of generalized trigonometric functions, and establish the Turán type inequalities of these functions.

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1. INTRODUCTION

For given complex numbers a, b and c with $c \neq 0, -1, -2, \dots$, the *Gaussian hypergeometric function* ${}_2F_1$ is the analytic continuation to the slit plane $\mathbb{C} \setminus [1, \infty)$ of the series

$$F(a, b; c; z) = {}_2F_1(a, b; c; z) = \sum_{n \geq 0} \frac{(a, n)(b, n)}{(c, n)} \frac{z^n}{n!}, \quad |z| < 1.$$

Here (a, n) is the Pochhammer symbol (rising factorial) $(\cdot, n) : \mathbb{C} \rightarrow \mathbb{C}$, defined by

$$(z, n) = \frac{\Gamma(z+n)}{\Gamma(z)} = \prod_{i=1}^n (z+i-1)$$

for $n \in \mathbb{Z}$, see [AS].

The integral representation of the hypergeometric function is given as follows [S, p. 20]

$$(1.1) \quad F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(c)(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt$$

$\operatorname{Re}(c) > \operatorname{Re}(a) > 0, |\arg(1-z)| < \pi$.

Special functions, such as the classical *gamma function* Γ , the *digamma function* ψ and the *beta function* $B(\cdot, \cdot)$ have close relation with hypergeometric function. For $x, y > 0$, these functions are defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

respectively.

Recently, Takeuchi [T] studied the (p, q) -trigonometric functions depending on two parameters. For $p = q$, these functions reduce to the so-called p -trigonometric functions introduced by Lindqvist in his highly cited paper [L]. In present, there

has been a vivid interest on the generalized trigonometric and hyperbolic functions, numerous papers have been published on the studies of generalized trigonometric functions and their inequalities, see, e.g., see [BBV, BBK, BS, BV1, BV2, EGL, JW, KVZ] and the references therein.

The following (p, q) -eigenvalue problem with Dirichlét boundary condition was considered by Drábek and Manásevich [DM]. Let $\phi_p(x) = |x|^{p-2}x$. For $T, \lambda > 0$ and $p, q > 1$

$$\begin{cases} (\phi_p(u'))' + \lambda \phi_q(u) = 0, & t \in (0, T), \\ u(0) = u(T) = 0. \end{cases}$$

They found the complete solution to this problem. The solution of this problem also appears in [T, Thm 2.1]. In particular, for $T = \pi_{p,q}$ the function $u(t) = \sin_{p,q}(t)$ is a solution to this problem with $\lambda = p/q(p-1)$, where

$$\pi_{p,q} = \int_0^1 (1-t^q)^{-1/p} dt = \frac{2}{q} B\left(1 - \frac{1}{p}, \frac{1}{q}\right).$$

For $p = q$, $\pi_{p,q}$ reduces to π_p , see, e.g., [BBV]. In order to give the definition of the function $\sin_{p,q}$, first we define its inverse function $\arcsin_{p,q}$, then the function itself. For $x \in [0, 1]$, set

$$F_{p,q}(x) = \arcsin_{p,q} = \int_0^x (1-t^q)^{-1/p} dt.$$

The function $F_{p,q} : [0, 1] \rightarrow [0, \pi_{p,q}/2]$ is an increasing homeomorphism, and

$$\sin_{p,q} = F_{p,q}^{-1},$$

is defined on the the interval $[0, \pi_{p,q}/2]$. The function $\sin_{p,q}$ can be extended to $[0, \pi_{p,q}]$ by

$$\sin_{p,q}(x) = \sin_{p,q}(\pi_{p,q} - x), \quad x \in [\pi_{p,q}/2, \pi_{p,q}].$$

By oddness, the further extension can be made to $[-\pi_{p,q}, \pi_{p,q}]$. Finally the functions $\sin_{p,q}$ is extended to whole \mathbb{R} by $2\pi_{p,q}$ -periodicity, see [EGL].

The generalized cosine function $\cos_{p,q}$ can be defined as

$$\cos_{p,q}(x) : \mathbb{R} \rightarrow \mathbb{R}, \quad x \in \mathbb{R}.$$

One can see easily that $\cos_{p,q}$ is even with period $2\pi_{p,q}$ and odd in about $\pi_{p,q}/2$. Setting $y = \sin_{p,q}(x)$ and letting $x \in [0, \pi_{p,q}/2]$, we get

$$\cos_{p,q}(x) = (1 - y^q)^{1/p} = (1 - \sin_{p,q}(x)^q)^{1/p}.$$

Clearly, $\cos_{p,q}$ is strictly decreasing with $\cos_{p,q}(0) = 1$ and $\cos_{p,q}(\pi_{p,q}/2) = 0$. From the above definition it follows that

$$(1.2) \quad |\cos_{p,q}(x)|^p + |\sin_{p,q}(x)|^q = 1, \quad x \in \mathbb{R}.$$

The generalized tangent function $\tan_{p,q}$ is defined by

$$\tan_{p,q}(x) = \frac{\sin_{p,q}(x)}{\cos_{p,q}(x)}, \quad x \in \{\mathbb{R} : x \neq (z + 1/2)\pi_{p,q}, z \in \mathbb{Z}\}.$$

The usual elementary trigonometric functions are the special case of these (p, q) -trigonometric functions when $p = q = 2$.

The differential p -arc length for a curve given by the parametric equations $x = x(t)$ and $y = y(t)$ can be computed as

$$ds = \sqrt[p]{\left(\frac{dx}{dt}\right)^p + \left(\frac{dy}{dt}\right)^p} dt.$$

Let's call an ellipse as a p -ellipse whose parametric equations are $x = a \cos_p(t)$ and $y = b \sin_p(t)$. The parameter P of the p -ellipse is determined by the integral

$$P = 4 \int_0^{\pi_p/2} \sqrt[p]{a^p \sin_p(t)^p + b^p \cos_p(t)^p} dt.$$

Applying the formula (1.2), we get

$$\begin{aligned} P &= 4 \int_0^{\pi_p/2} \sqrt[p]{a^p(1 - \cos_p(t)^p) + b^p \cos_p(t)^p} dt \\ &= 4 \int_0^{\pi_p/2} \sqrt[p]{a^p - (a^p - b^p) \cos_p(t)^p} dt. \end{aligned}$$

Letting $r^p = (1 - b^p/a^p)$, we can write

$$(1.3) \quad P = 4a \int_0^{\pi_p/2} \sqrt[p]{1 - r^p \cos_p(t)^p} dt.$$

Since $\cos_p(t) = \sin_p(\pi_p/2 - t)$, so we can write for any function $j(x)$

$$\int_0^{\pi_p/2} j(\cos_p(t)) dt = \int_0^{\pi_p/2} j\left(\sin_p\left(\frac{\pi_p}{2} - t\right)\right) dt.$$

Substituting $\theta = \pi_p/2 - t$, we write this as

$$\int_0^{\pi_p/2} j(\cos_p(t)) dt = \int_0^{\pi_p/2} j(\sin_p(\theta)) d\theta.$$

Therefore we replace \cos_p by \sin_p in (1.3), and obtain

$$P = 4a \int_0^{\pi_p/2} \sqrt[p]{1 - r^p \sin_p(t)^p} dt.$$

This way we expressed the perimeter in terms of the following function

$$(1.4) \quad E_p(r) = \int_0^{\pi_p/2} \sqrt[p]{1 - r^p \sin_p(t)^p} dt$$

which is called p -generalized elliptic integral of the first kind. Similarly, we defined the (p, q) -generalized elliptic integral of the first kind by

$$(1.5) \quad E_{p,q}(r) = \int_0^{\pi_{p,q}/2} \sqrt[p]{1 - r^q \sin_p(t)^q} dt.$$

Substituting $x = \sin_{p,q}(t)$ and applying (1.2), the formula (1.7) can be written as

$$(1.6) \quad E_{pq}(r) = \int_0^1 (1 - x^q)^{-1/p} (1 - x^q r^q)^{1/p} dx.$$

For $p, q > 1$, $x \in [0, 1]$ and $r \in [0, 1]$, we define the function $\arcsn_{pq} : [0, 1] \rightarrow [0, K_{pq}]$ by

$$\arcsn_{pq}(x) = \arcsn_{pq}(x, r) = \int_0^x \frac{dt}{\sqrt[p]{(1 - t^q)(1 - r^q t^q)}},$$

and call it generalized inverse Jacobian elliptic function, where K_{pq} is called (p, q) -generalized elliptic integral of the first kind, and defined as

$$(1.7) \quad K_{pq}(r) = \int_0^1 \frac{dt}{\sqrt[p]{(1 - t^q)(1 - r^q t^q)}} = \arcsn_{pq}(1, r).$$

Substituting $t = \sin_{p,q}(\theta)$ in the above formula, we get

$$K_{p,q}(r) = \int_0^{\pi_{p,q}/2} \frac{d\theta}{\sqrt[p]{1 - r^q \sin_{p,q}(\theta)^q}}.$$

Clearly, $\arcsn_{pq}(x, r)$ is strictly increasing in x , and its inverse $\sn_{pq} : [0, K_{pq}] \rightarrow [0, 1]$ is also strictly increasing, and called generalized Jacobian elliptic function [T].

Letting $t = x^{1/q}$ in (1.7) and utilizing the formula (1.1), we get

$$\begin{aligned} K_{pq}(r) &= \frac{1}{q} \int_0^1 x^{1/q-1} (1 - x)^{-1/p} (1 - x r^q)^{-1/p} dx \\ &= \frac{1}{q} \frac{B(1/q, 1 - 1/p + 1/q - 1/q)}{B(1/q, 1 - 1/p)} \int_0^1 x^{1/q-1} (1 - x)^{1-1/p+1/q-1/q-1} (1 - x r^q)^{-1/p} dx \\ &= \frac{1}{q} B\left(\frac{1}{q}, 1 - \frac{1}{q}\right) F\left(\frac{1}{p}, \frac{1}{q}; 1 - \frac{1}{p} + \frac{1}{q}; r^q\right) \\ &= \frac{\pi_{p,q}}{2} F\left(\frac{1}{p}, \frac{1}{q}; 1 - \frac{1}{p} + \frac{1}{q}; r^q\right). \end{aligned}$$

Similarly, by applying the formulas (1.6) and (1.1), the function $E_{p,q}(r)$ can be expressed in terms of hypergeometric function as below

$$E_{pq}(r) = \frac{\pi_{p,q}}{2} F\left(-\frac{1}{p}, \frac{1}{q}; 1 - \frac{1}{p} + \frac{1}{q}; r^q\right).$$

For the convenience of the reader and easy reference we recall from the above formulas that for $p, q > 1$ and $r \in (0, 1)$, the (p, q) -generalized elliptic integrals of the first and second kind are defined as

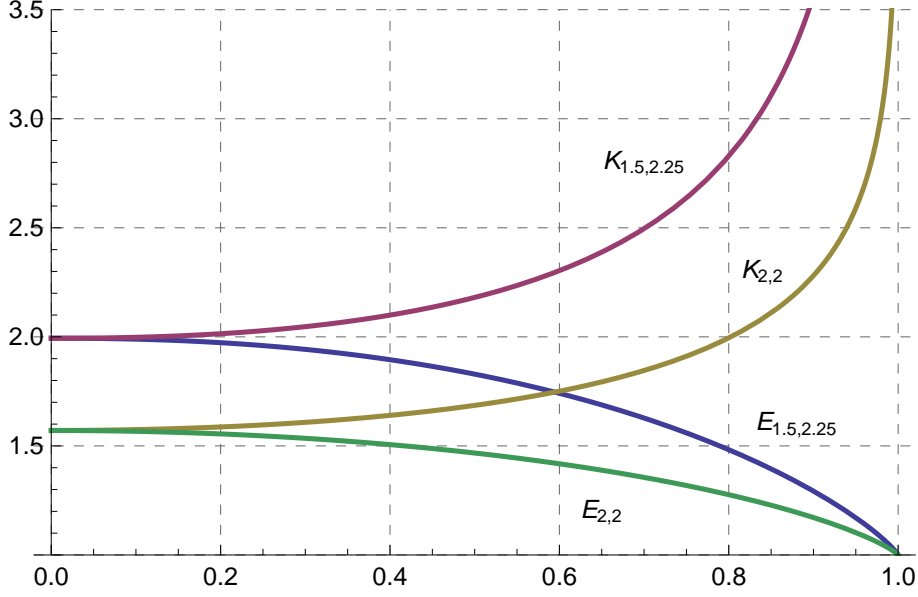


FIGURE 1. Graphs of $K_{1.5,2.25}(r)$, $E_{1.5,2.25}(r)$, $K_{2,2}(r) = \mathcal{K}(r)$ and $E_{2,2}(r) = \mathcal{E}(r)$, with $K_{1.5,2.25}(1) = E_{1.5,2.25}(1) = \pi_{1.5,2.25}/2 \approx 1.9937$.

$$(1.8) \quad \begin{cases} K_{p,q}(r) = \int_0^{\pi_{p,q}/2} \frac{dt}{\sqrt[p]{1 - r^q \sin_p(t)^q}} = \int_0^1 \frac{dt}{\sqrt[p]{(1-t^q)(1-r^q t^q)}} \\ E_{p,q}(r) = \int_0^{\pi_{p,q}/2} \sqrt[p]{1 - r^q \sin_p(t)^q} dt = \int_0^1 \sqrt[p]{\frac{1-r^q t^q}{1-t^q}} dt \\ K_{p,q}(0) = \frac{\pi_{p,q}}{2} = E_{p,q}(0), \quad K_{p,q}(1) = \infty, \quad E_{p,q}(1) = 0. \end{cases}$$

For $p = q$, we denote

$$K_p(r) = K_{pp}(r) = \frac{\pi_p}{2} F\left(\frac{1}{p}, \frac{1}{p}; 1; r^p\right),$$

and

$$E_p(r) = E_{pp}(r) = \frac{\pi_p}{2} F\left(-\frac{1}{p}, \frac{1}{p}; 1; r^p\right).$$

Obviously $K_p = \mathcal{K}$ and $E_p = \mathcal{E}$ for $p = 2$, where \mathcal{K} and \mathcal{E} are the classical elliptic integrals of the first and second kind, respectively. We refer to reader to see the Book [AVV] for the history and the large study on these functions. The generalization and the inequalities of elliptic integrals were studied by numerous authors after the publication of the landmark paper [BBG]. The generalized elliptic integrals of the first and second kind on $(0, 1)$ are defined respectively by

$$\mathcal{K}_{a,b,c}(r) = \frac{B(a,b)}{2} F(a, b; c; r^2),$$

$$\mathcal{E}_{a,b,c}(r) = \frac{B(a,b)}{2} F(a-1, b; c; r^2),$$

for $0 < a < \min\{c, 1\}$ and $0 < b < c \leq a + b$, see [AQVV]. Clearly, $\mathcal{K}_{1/2,1/2,1} = \mathcal{K}$, $\mathcal{E}_{1/2,1/2,1} = \mathcal{E}$. For the monotonic properties and the inequalities of these functions, see [AQVV, HLTV, HVV]. For the historical background and study about the classical and generalized case of elliptic integrals, we refer to reader to see, e.g., [AQVV, AQ, AVV, B1, B3, BV3, HLTV, HVV, WZC], and the bibliography of these papers.

Before we present the main results of this paper we recall some definitions as follows:

A function $f: (0, \infty) \rightarrow (0, \infty)$ is said to be logarithmically convex, or log-convex, if its natural logarithm $\ln f$ is convex, that is, for all $x, y > 0$ and $\lambda \in [0, 1]$ we have

$$f(\lambda x + (1 - \lambda)y) \leq [f(x)]^\lambda [f(y)]^{1-\lambda}.$$

The function f is log-concave if the above inequality is reversed.

A function $g: (0, \infty) \rightarrow (0, \infty)$ is said to be geometrically (or multiplicatively) convex if it is convex with respect to the geometric mean, that is, if for all $x, y > 0$ and all $\lambda \in [0, 1]$ the inequality

$$g(x^\lambda y^{1-\lambda}) \leq [g(x)]^\lambda [g(y)]^{1-\lambda}$$

holds. The function g is called geometrically concave if the above inequality is reversed. We also note that the differentiable function f is log-convex (log-concave) if and only if $x \mapsto f'(x)/f(x)$ is increasing (decreasing), while the differentiable function g is geometrically convex (concave) if and only if the function $x \mapsto xg'(x)/g(x)$ is increasing (decreasing), for more details see [B2].

This paper consists of three sections. In the first section, we give the introduction and define the definitions of the functions which are being studied in the paper. In the second section, we state our main results. The third section contains few lemmas and the proof of the main result.

2. MAIN RESULT

The main result of this paper reads as follows.

2.1. Theorem. *For $p, q > 1$, $r \in (0, 1)$ and $r' = (1 - r^q)^{1/q}$, we have*

$$(2.2) \quad \frac{d}{dr} K_{p,q}(r) = \frac{1}{r (r')^q} (E_{p,q}(r) - (r')^q K_{p,q}(r))$$

$$(2.3) \quad \frac{d}{dr} (E_{p,q}(r)) = \frac{q}{pr} (E_{p,q}(r) - K_{p,q}(r))$$

$$(2.4) \quad \frac{d^2}{dr^2} K_{p,q}(r) = \left(\frac{q (r')^q}{p} + qr^q - 2 (r')^q \right) E_{p,q}(r) + \left(2 (r')^{2q} - \frac{q}{p} (r')^q \right) K_{p,q}(r)$$

$$(2.5) \quad \frac{d^2}{dr^2} E_{p,q}(r) = \frac{q}{pr^2} \left(\left(\frac{q}{p} - \frac{1}{(r')^q} - 1 \right) E_{p,q}(r) + \left(2 - \frac{q}{p} \right) K_{p,q}(r) \right).$$

2.6. Theorem. For $p, q > 1$ and $r \in (0, 1)$, we have

- (1) the function $r \mapsto K_{p,q}(r)$ is strictly increasing and log-convex. Moreover, $r \mapsto K_{p,q}(r)$ is strictly geometrically convex on $(0, 1)$.
- (2) The function $r \mapsto E_{p,q}(r)$ is strictly decreasing and geometrically concave on $(0, 1)$.

2.7. Theorem. For fixed $r \in (0, 1)$ and $q > 0$,

- (1) the function $p \mapsto K_{p,q}(r)$ is strictly increasing and log-concave on $(0, \infty)$,
 - (2) the function $p \mapsto E_{p,q}(r)$ is strictly increasing and log-concave on $(0, \infty)$.
- For fixed $r \in (0, 1)$ and $p > 0$,
- (3) the function $q \mapsto K_{p,q}(r)$ is strictly decreasing and log-convex on $(0, \infty)$,
 - (4) the function $q \mapsto E_{p,q}(r)$ is strictly decreasing and log-convex on $(0, \infty)$.

In particular, for $r \in (0, 1)$, the following Turán type inequalities hold true

$$K_{p,q}(r)^2 \geq K_{p-1,q}(r)K_{p+1,q}(r), \quad p > 1, q > 0,$$

$$E_{p,q}(r)^2 \geq E_{p-1,q}(r)E_{p+1,q}(r), \quad p > 1, q > 0,$$

$$K_{p,q}(r)^2 \leq K_{p,q-1}(r)K_{p,q+1}(r), \quad p > 0, q > 1,$$

$$E_{p,q}(r)^2 \leq E_{p,q-1}(r)E_{p,q+1}(r), \quad p > 0, q > 1.$$

The following corollary is the especial case of the above theorem.

2.8. Corollary. For $r \in (0, 1)$, we have the following double inequalities,

$$\sqrt{K_{1,2}(r)K_{3,2}(r)} \leq \mathcal{K}(r) \leq \sqrt{K_{2,1}(r)K_{2,3}(r)},$$

$$\sqrt{E_{1,2}(r)E_{3,2}(r)} \leq \mathcal{E}(r) \leq \sqrt{E_{2,1}(r)E_{2,3}(r)}.$$

2.9. Theorem. For $p, q > 1$ and $r \in (0, 1)$, $\lambda < \frac{1}{2}$, we have

$$(2.10) \quad K_{p,q}(r) = \frac{\pi_{p,q}}{2} \sum_{n=0}^{\infty} \binom{\frac{1}{p}-1}{n} \frac{1}{(1-\lambda)^{n+1-\frac{1}{p}}} \sum_{j=0}^n (-1)^j \binom{n}{j} \binom{-\frac{1}{q}}{j} \binom{\frac{1}{p}-1-\frac{1}{q}}{j} (\lambda)^{n-j} r^{qj}.$$

$$(2.11) \quad E_{p,q}(r) = \frac{\pi_{p,q}}{2} \sum_{n=0}^{\infty} \binom{\frac{1}{p}}{n} \frac{1}{(1-\lambda)^{n-\frac{1}{p}}} \sum_{j=0}^n (-1)^j \binom{n}{j} \binom{-\frac{1}{q}}{j} \binom{\frac{1}{p}-1-\frac{1}{q}}{n} (\lambda)^{n-j} r^{qn}.$$

2.12. Remark. If we let $\lambda = 0$, then (2.10) becomes

$$\begin{aligned} K_{p,q}(r) &= \frac{\pi_{p,q}}{2} \sum_{n=0}^{\infty} \binom{\frac{1}{p}-1}{n} \binom{n}{n} (-1)^n \binom{-\frac{1}{q}}{n} \binom{\frac{1}{p}-1-\frac{1}{q}}{n} r^{qn} \\ &= \frac{\pi_{p,q}}{2} \sum_{n=0}^{\infty} \frac{\left(1-\frac{1}{p}\right)_n \left(\frac{1}{q}\right)_n \left(1+\frac{1}{q}-\frac{1}{p}\right)_n}{(1)_n (1)_n} \frac{r^{qn}}{n!} \\ &= \frac{\pi_{p,q}}{2} {}_3F_2 \left(1-\frac{1}{p}, \frac{1}{q}, 1+\frac{1}{q}-\frac{1}{p}; 1, 1; r^q \right). \end{aligned}$$

Similary, from (2.11) we get

$$\begin{aligned} E_{p,q}(r) &= \frac{\pi_{p,q}}{2} \sum_{n=0}^{\infty} \binom{-\frac{1}{p}}{n} \binom{n}{n} (-1)^n \binom{-\frac{1}{q}}{n} \binom{\frac{1}{p}-1-\frac{1}{q}}{n} r^{qn} \\ &= \frac{\pi_{p,q}}{2} \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{p}\right)_n \left(\frac{1}{q}\right)_n \left(1+\frac{1}{q}-\frac{1}{p}\right)_n}{(1)_n (1)_n} \frac{r^{qn}}{n!} \\ &= \frac{\pi_{p,q}}{2} {}_3F_2 \left(-\frac{1}{p}, \frac{1}{q}, 1+\frac{1}{q}-\frac{1}{p}; 1, 1; r^q \right), \end{aligned}$$

by letting $\lambda = 0$.

3. PRELIMINARIES AND PROOFS

Before quote here few lemmas which will be used in the proof of theorems. The following lemma follows easily from the definition and (1.2).

3.1. Lemma. [EGL, Proposition 3.1] *For all $x \in [0, \pi_{p,q}/2]$,*

- (1) $\frac{d}{dx} \sin_{p,q} x = \cos_{p,q} x,$
- (2) $\frac{d}{dx} \cos_{p,q} x = -\frac{q}{p} (\cos_{p,q} x)^{2-p} (\sin_{p,q} x)^{q-1},$
- (3) $\frac{d}{dx} (-\cos_{p,q} x)^{p-1} = \frac{(p-1)q}{p} (\sin_{p,q} x)^{q-1},$
- (4) $\frac{d}{dx} (\sin_{p,q} x)^q = q (\sin_{p,q} x)^{q-1} \cos_{p,q} x.$

3.2. Lemma. [BBV, Lemma 2] *If the function $v \mapsto K(v, t)$ is positive and (strictly) geometrically convex on $[a, b]$ for $t \in (0, x)$, with $0 < a < b$ and $x > 0$. Then the function*

$$v \mapsto f_v(x) = \int_0^x K(v, t) dt$$

is also (strictly) geometrically convex.

3.3. Lemma ([AR, Theorem (The λ -method)]). *Suppose that the function G given by $G(x) = \frac{g(x)}{(1-\alpha x^\eta)^\xi}$ satisfies $g, G \in L^1[0, 1]$ where $0 < \alpha \leq 1, \eta > 0, \lambda < \frac{1}{2}, \xi \in \mathbb{R}$, and*

$$b_j = b_j(\alpha, \eta) = \alpha^j \int_0^1 t^{j\eta} g(t) dt,$$

it follows that

$$(3.4) \quad \int_0^1 \frac{g(x)}{(1-\alpha x^\eta)^\xi} dx = \sum_{n=0}^{\infty} \frac{(\xi)_n}{n!(1-\lambda)^{n+\xi}} \sum_{j=0}^n \binom{n}{j} (-\lambda)^{n-j} b_j(\alpha, \eta).$$

Proof of Theorem 2.1. Applying the derivative formulas given in Lemma 3.1 and utilizing the identity (1.2), we get

$$(3.5) \quad \frac{d}{dx} \left(\frac{-(\cos_{p,q} x)^{p-1}}{(1-k^q (\sin_{p,q} x)^q)^{1-\frac{1}{p}}} \right) = \frac{(p-1)q}{p} \frac{(\sin_{p,q} x)^{q-1} (1-k^q)}{(1-k^q (\sin_{p,q} x)^q)^{2-\frac{1}{p}}},$$

Now by using the definition and (3.5), we have

$$\begin{aligned} \frac{d}{dr} K_{p,q}(k) &= \int_0^{\frac{\pi_{p,q}}{2}} \frac{r^q}{(r')^q} \sin_{p,q} x \frac{d}{dx} \left(\frac{-(\cos_{p,q} x)^{p-1}}{(1-r^q (\sin_{p,q} x)^q)^{1-\frac{1}{p}}} \right) dx \\ &= \frac{r^q}{(r')^q} \int_0^{\frac{\pi_{p,q}}{2}} \frac{(\cos_{p,q} x)^p}{(1-r^q (\sin_{p,q} x)^q)^{1-\frac{1}{p}}} dx \\ &= \frac{k^q}{(r')^q} \int_0^{\frac{\pi_{p,q}}{2}} \frac{1 - (\sin_{p,q} x)^q}{(1-r^q (\sin_{p,q} x)^q)^{1-\frac{1}{p}}} dx \\ &= \frac{r^{q-1}}{(r')^q} \int_0^{\frac{\pi_{p,q}}{2}} \frac{1 - r^q (\sin_{p,q} x)^q + r^q - 1}{(1-r^q (\sin_{p,q} x)^q)^{1-\frac{1}{p}}} dx \\ &= \frac{1}{r (r')^q} (E_{p,q}(r) - (r')^q K_{p,q}(r)). \end{aligned}$$

Similarly, we get

$$\begin{aligned} \frac{d}{dr} E_{p,q}(r) &= \int_0^{\frac{\pi_{p,q}}{2}} \frac{1}{p} (1-r^q (\sin_{p,q} x)^q)^{\frac{1}{p}-1} (-q) r^{q-1} (\sin_{p,q} x)^q dx \\ &= \frac{q}{pr} \left[\int_0^{\frac{\pi_{p,q}}{2}} (1-r^q (\sin_{p,q} x)^q)^{\frac{1}{p}} dx - \int_0^{\frac{\pi_{p,q}}{2}} (1-r^q (\sin_{p,q} x)^q)^{\frac{1}{p}-1} dx \right] \\ &= \frac{q}{pr} (E_{p,q}(r) - K_{p,q}(r)). \end{aligned}$$

□

With the use of formula (2.2) and (2.3), a lengthy and trivial computation yields the proof of (2.4) and (2.5). □

For $p, q > 1$ and $r \in (0, 1)$, it is easy to observe that the functions $K_{p,q}(r)$ and $E_{p,q}(r)$ satisfy the following hypergeometric differential equations

$$(3.6) \quad \frac{d^2}{dr^2} K_{p,q}(r) - \left(\frac{q(r')^q}{p} + qr^q - 2(r')^q \right) E_{p,q}(r) - \left(2(r')^{2q} - \frac{q}{p} (r')^q \right) K_{p,q}(r) = 0,$$

$$(3.7) \quad \frac{d^2}{dr^2} (E_{p,q}(r)) + \frac{1}{r} \left(2 - \frac{q}{p} \right) \frac{d}{dr} (E_{p,q}(r)) + \frac{q}{p} \frac{r^{q-2}}{1 - r^q} E_{p,q}(r) = 0,$$

respectively.

Proof of Theorem 2.6. We define

$$f(r) = (1 - r^q \sin_{p,q}(x))$$

for $p, q > 1$, $r \in (0, 1)$ and $x \in (0, \pi_{p,q}/2)$. Differentiating with respect to r , we get

$$(\log f(r))' = \left(1 - \frac{1}{p} \right) \sin_{p,q}(x)^q \frac{r^{q-1}}{1 - r^q \sin_{p,q}(x)^q} > 0,$$

$$(\log f(r))'' = \left(1 - \frac{1}{p} \right) \sin_{p,q}(x)^q \frac{(q-1)r^{q-1} + r^2(q-1) \sin_{p,q}(x)^q}{(1 - r^q \sin_{p,q}(x)^q)^2} > 0.$$

But using the fact that the integral preserves the monotonicity and log-convexity, This implies that for $p, q > 1$ and $x \in (0, \pi_{p,q}/2)$ the function $r \mapsto K_{p,q}(r)$ is strictly increasing and log-convex on $(0, 1)$.

For the proof of geometrical convexity, by simple computation we get

$$f'(r) = \left(1 - \frac{1}{p} \right) \sin_{p,q}(x)^q r^{q-1} (1 - r^q \sin_{p,q}(x)^q)^{1/p-2} > 0,$$

and

$$\begin{aligned} \left(\frac{r f'(r)}{f(r)} \right)' &= \left(1 - \frac{1}{p} \right) q \sin_{p,q}(x)^q \left(\frac{r^q (1 - r^q \sin_{p,q}(x)^q)^{1/p-2}}{1 - r^q \sin_{p,q}(x)^q} \right)' \\ &= \left(1 - \frac{1}{p} \right) q^2 \sin_{p,q}(x)^q \frac{r^{q-1}}{(1 - r^q \sin_{p,q}(x)^q)^2} > 0. \end{aligned}$$

Applying Lemma 3.2, we obtain that the function $r \mapsto K_{p,q}(r)$ is strictly geometrically convex on $(0, 1)$.

For the proof of part (2), let

$$g(r) = (1 - r^q \sin_{p,q}(x)^q)^{1/q},$$

for $p, q > 1$, $r \in (0, 1)$ and $x \in (0, \pi_{p,q}/2)$. A simple computation yields

$$(\log(g(r)))' = -\frac{q}{p} \frac{r^{q-1} \sin_{p,q}(x)^q}{1 - r^q \sin_{p,q}(x)^q} < 0,$$

and

$$\left(\frac{r g'(r)}{g(r)} \right)' = -\frac{q}{p} \sin_{p,q}(x)^q \frac{qr^{q-1} (1 - r^q \sin_{p,q}(x)^q) - r^q (-qr^{q-1} \sin_{p,q}(x)^q)}{(1 - r^q \sin_{p,q}(x)^q)^2}$$

$$= -\frac{q^2}{p} \sin_{p,q}(x)^q \frac{r^{q-1}}{(1 - r^q \sin_{p,q}(x))^2} < 0.$$

Now the rest of proof follows immediately from Lemma 3.2. \square

Proof of Theorem 2.7. For $p, q > 0$, we define

$$f(p, q) = (1 - t^q)^{-1/p} (1 - r^q t^q)^{-1/p}$$

and

$$g(p, q) = (1 - t^q)^{-1/p} (1 - r^q t^q)^{1/p}.$$

An easy computation yields that

$$\begin{aligned} \frac{\partial}{\partial p}(\log f(p, q)) &= \frac{1}{p^2} \log((1 - t^q)(1 - r^q t^q)) > 0, \\ \frac{\partial^2}{\partial p^2}(\log f(p, q)) &= -\frac{2}{p^3} \log((1 - t^q)(1 - r^q t^q)) < 0, \\ \frac{\partial}{\partial p}(\log g(p, q)) &= \frac{1}{p^2} \log\left(\frac{1 - t^q}{1 - r^q t^q}\right) > 0, \\ \frac{\partial^2}{\partial p^2}(\log g(p, q)) &= -\frac{2}{p^3} \log\left(\frac{1 - t^q}{1 - r^q t^q}\right) < 0. \end{aligned}$$

For fixed $q > 0$, the functions $p \mapsto f(p, q)$ and $p \mapsto g(p, q)$ are strictly increasing and log-concave on $(0, \infty)$. By using the fact that integral preserves the monotonicity and log-concavity. It follows that for fixed $q > 0$ and $r \in (0, 1)$ the functions $p \mapsto K_{p,q}(r)$ and $p \mapsto E_{p,q}(r)$ are strictly increasing and log-concave on $(0, \infty)$. For the proof of part (3), we get

$$\begin{aligned} \frac{\partial}{\partial q}(\log f(p, q)) &= \frac{r^q(\log(r)(1 + t^q - 2r^q t^q) + t^q \log(t)(1 - r^q))}{p(1 - r^q)(1 - r^q t^q)} < 0, \\ \frac{\partial^2}{\partial q^2}(\log f(p, q)) &= \frac{1}{p} \left(\frac{(\log(r) + \log(t))^2}{(1 - r^q t^q)^2} - \frac{(\log(r) + \log(t))^2}{1 - r^q t^q} + \frac{r^q \log^2(r)}{(1 - r^q)^2} \right) > 0. \end{aligned}$$

This implies that for fixed $p > 0$ and $r \in (0, 1)$, the function $q \mapsto f(p, q)$ is strictly decreasing and log-convex on $(0, \infty)$.

In order to prove the monotonicity of the function $q \mapsto g(p, q)$ given in part (4), first we show that for $t \in (0, 1)$ and $q > 0$ the function

$$h_q(t) = \left(\frac{t^q}{1 - t^q} \right) \log\left(\frac{1}{t}\right)$$

has the derivative with respect to t

$$(h_q(t))' = \frac{t^{q-1}(\log(1/t^q) + t^q - 1)}{(1 - t^q)^2},$$

which is positive by the inequality $\log(x) > 1 - 1/x$, $x > 1$. Since, the function $h_q(t)$ is strictly increasing in $t \in (0, 1)$. Partially differentiating $g(p, q)$ with respect q , we get

$$\frac{\partial}{\partial q}(\log g(p, q)) = \frac{1}{p} (h_q(rt) - h_q(t)),$$

which is negative because the function $h_q(t)$ is strictly increasing in $t \in (0, 1)$. Hence $g(p, q)$ is strictly decreasing on $(0, \infty)$.

For proving the log-convexity of $q \mapsto g(p, q)$, we get

$$\frac{\partial^2}{\partial q^2}(\log g(p, q)) = \frac{1}{p^2} (j_q(t) - j_q(rt)),$$

where $j_q(t) = \left(\frac{t^q}{(1-t^q)^2} \right) \log(t)^2$. Now it is enough to prove that $j_q(t), t \in (0, 1)$ is strictly increasing and positive. Writing $k_q(t^q) = 2 - 2t^q + (1 + t^q) \log(t^q)$, we get

$$j_q(t)' = \frac{t^{q-1} \log(t) k_q(t^q)}{(1 - t^q)^3},$$

which is positive, because the function $k_q(t)$ is strictly increasing in t and negative with $k_p(0) = 0$, in fact

$$k_q(t)' = \log(t) + 1/t - 1 > 0.$$

So far we have proved that the function $q \mapsto g(p, q)$ is strictly decreasing and log-convex on $(0, \infty)$. By repeating the same argument from the proof of part (1) and (2), that the integral preserves the monotonicity and log-convexity, we conclude that for fixed $p > 0$ and $r \in (0, 1)$ the function $q \mapsto E_{p,q}(r)$ is strictly decreasing and log-convex on $(0, \infty)$. This completes the proof. \square

Proof of Theorem 2.9. Let $\alpha = r^q, \eta = q, \xi = 1 - \frac{1}{p}, g(x) = (1 - x^q)^{-\frac{1}{p}}$. Applying formula (3.4) and

$$\frac{\left(\frac{1}{q}\right)_j}{j!} = (-1)^j \binom{-\frac{1}{q}}{j}, \quad \frac{\left(\frac{1}{q} + 1 - \frac{1}{p}\right)_j}{j!} = (-1)^j \binom{\frac{1}{p} - 1 - \frac{1}{q}}{j},$$

we get

$$\begin{aligned} b_j(\alpha, \eta) &= r^{qj} \int_0^1 t^{jq} (1 - t^q)^{-\frac{1}{p}} dt \\ &= r^{qj} \frac{1}{q} \int_0^1 u^j (1 - u)^{-\frac{1}{p}} u^{\frac{1}{q}-1} du \\ &= \frac{r^{qj}}{q} B\left(\frac{1}{q} + j, 1 - \frac{1}{p}\right) \\ &= \frac{\pi_{p,q} r^{qj}}{2} \binom{-\frac{1}{q}}{j} \binom{\frac{1}{p} - 1 - \frac{1}{q}}{j}. \end{aligned}$$

Now the claim follows easily if we apply the formula

$$\frac{\left(1 - \frac{1}{p}\right)_n}{n!} = (-1)^n \binom{\frac{1}{p} - 1}{n}.$$

This completes the proof of (2.10). The proof of (2.11) follows similarly. \square

REFERENCES

- [AS] M. ABRAMOWITZ, I. STEGUN, EDS.: *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, National Bureau of Standards, Dover, New York, 1965.
- [AQ] H. ALZER AND S.-L. QIU: *Monotonicity theorems and inequalities for the complete elliptic integrals*. J. Comput. Appl. Math. 172 (2004), no. 2, 289–312.
- [AQVV] G. D. ANDERSON, S.-L. QIU, M. K. VAMANAMURTHY, AND M. VUORINEN: *Generalized elliptic integrals and modular equation*. Pacific J. Math. Vol. 192, No. 1, (2000), 1–37.
- [AR] H. ALZER AND K. C. RICHARD: *Series representation for special functions and mathematical constants*, Ramanujan J., 2015 (to appear).
- [AVV] G.D. ANDERSON, M.K. VAMANAMURTHY, M. VUORINEN: *Conformal Invariants, Inequalities and Quasiconformal Maps*, J. Wiley, 1997.
- [B1] Á. BARICZ: *Turán type inequalities for generalized complete elliptic integrals*. Math. Z. 256(4), (2007), 895–911.
- [B2] Á. BARICZ: *Geometrically concave univariate distributions*, J. Math. Anal. Appl. 363(1), (2010), 182–196.
- [B3] Á. BARICZ: *Turán type inequalities for hypergeometric functions*, Proc. Amer. Math. Soc. 136 (2008), no. 9, 3223–3229.
- [BBT] Á. BARICZ, B.A. BHAYO, T.K. POGÁNY: *Functional inequalities for generalized inverse trigonometric and hyperbolic functions*, J. Math. Anal. Appl. 417 (2014), 244–259.
- [BBV] Á. BARICZ, B.A. BHAYO, M. VUORINEN: *Turán type inequalities for generalized inverse trigonometric functions*, Filomat 29:2 (2015), 303–313, <http://arxiv.org/abs/1209.1696>.
- [BBK] Á. BARICZ, B. A. BHAYO, R. KLÉN: *Convexity properties of generalized trigonometric and hyperbolic functions*, Aequat. Math. 89 (2015), 473–484.
- [Be] V. BELEVITCH: *The Gauss hypergeometric ratio as a positive real function*, SIAM J. Math. Anal. 13(6) (1982), 1024–1040.
- [BBG] B. C. BERNDT, S. BHARGAVA, AND F. G. GARVAN: *Ramanujan’s theories of elliptic functions to alternative bases*. Trans. Amer. Math. Soc. 347 (1995), no. 11, 4163–4244.
- [BS] B.A. BHAYO, J. SÁNDOR: *Inequalities connecting generalized trigonometric functions with their inverses*, Issues of Analysis 2(20) (2013) 82–90.
- [BV1] B.A. BHAYO, M. VUORINEN: *On generalized trigonometric functions with two parameters*, J. Approx. Theory 164 (2012), 1415–1426.
- [BV2] B.A. BHAYO, M. VUORINEN: *Inequalities for eigenfunctions of the p -Laplacian*, Issues of Analysis 2 (20) (2013), 13–35.
- [BV3] B.A. BHAYO, M. VUORINEN: *On generalized complete elliptic integrals and modular functions*, Proc. Edinb. Math. Soc. (2012) 55, 591–611.
- [BE] P.J. BUSHILL, D.E. EDMUNDS: *Remarks on generalised trigonometric functions*, Rocky Mountain J. Math. 42 (2012), 13–52.
- [EGL] D.E. EDMUNDS, P. GURKA, J. LANG: *Properties of generalized trigonometric functions*, J. Approx. Theory 164 (2012), 47–56.
- [DM] P. DRÁBEK, R. MANÁSEVICH: *On the closed solution to some p -Laplacian nonhomogeneous eigenvalue problems*, Diff. and Int. Eqns. 12 (1999), 723–740.

- [EMOT] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER, F.G. TRICOMI: *Higher Transcendental Functions*, vol. I, Melbourne, 1981.
- [HLVV] V. HEIKKALA, H. LINDÉN, M. K. VAMANAMURTHY, AND M. VUORINEN: *Generalized elliptic integrals and the Legendre M-function*. J. Math. Anal. Appl. 338 (2008), 223–243, arXiv:math/0701438.
- [HVV] V. HEIKKALA, M. K. VAMANAMURTHY, AND M. VUORINEN: *Generalized elliptic integrals*. Comput. Methods Funct. Theory 9 (2009), no. 1, 75–109, arXiv:math/0701436.
- [JW] W.-D. JIANG, M.-K. WANG, Y.-M. CHU, Y.-P. JIANG, F. QI: *Convexity of the generalized sine function and the generalized hyperbolic sine function*, J. Approx. Theory 174 (2013), 1–9.
- [KP] D.B. KARP AND E.G. PRILEPKINA: *Parameter convexity and concavity of generalized trigonometric functions*, J. Math. Anal. Appl. Vol. 421, Issue 1, 1 (2015), 370–382.
- [KVZ] R. KLÉN, M. VUORINEN, X.-H. ZHANG: *Inequalities for the generalized trigonometric and hyperbolic functions*, J. Math. Anal. Appl. 409 (2014), 521–529.
- [L] P. LINDQVIST: *Some remarkable sine and cosine functions*, Ricerche di Matematica Vol. XLIV (1995) 269–290.
- [LP] P. LINDQVIST, J. PEETRE: *p-arclength of the q-circle*, The Mathematics Student 72 (2003) 139–145.
- [Mi] D.S. MITRINOVIĆ: *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.
- [Ne] E. NEUMAN: *Inequalities and bounds for generalized complete elliptic integrals*, J. Math. Anal. Appl. 373 (2011), 203–213.
- [O] F.W.J. OLVER, D.W. LOZIER, R.F. BOISVERT, C.W. CLARK, EDS.: *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.
- [SV] S. SIMIĆ, M. VUORINEN: *Landen inequalities for zero-balanced hypergeometric functions*, Abstr. Appl. Anal. 2012 (2012) Art. 932061.
- [S] L.J. SLATER: *Generalized Hypergeometric Functions*. Cambridge University Press, Cambridge, 1966.
- [T] S. TAKEUCHI: *Generalized Jacobian elliptic functions and their application to bifurcation problems associated with p-Laplacian*, J. Math. Anal. Appl. 385 (2012), 24–35.
- [WZC] G. WANG, X. ZHANG, AND Y. CHU: *Inequalities for the generalized elliptic integrals and modular functions*. J. Math. Anal. Appl. 331 (2007), no. 2, 1275–1283.

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